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# Euler's And Hamilton's Path Through One-Dimension Space

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**Abstract:** This article will present a technique for traversing a graph. Several questions arise here. For example, can we walk along the edges of a graph starting from a vertex and returning to it by visiting each edge of the graph exactly once? Similarly, can we walk along the edges of a graph starting from a vertex and returning to it while visiting each vertex of the graph exactly once? As can be seen, both questions are identical, but what is important is to consider two circuits that answer the above questions, namely the Euler circuit and the Hamilton circuit. Solving the Hamilton circuit for most graphs is very difficult. In this section, we will examine these questions and discuss the difficulty of solving them.

**Keywords:** Graph, Euler's Path, Hamilton's Paths, Analysis of path circuit.

**Introduction:** To better understand the path, we must first have a brief introduction to graphs first. Graphs are discrete structures consisting of vertices and the edges that connect these vertices. There are different types of graphs, depending on whether the edges are directed, whether multiple edges can connect a pair of vertices, and whether the graphs are cyclic. Traversing on the graph edges and crossing the vertices a path are created. Given the importance of these two topics, Euler and Hamilton paths are one of the fundamental concepts in graph theory, a branch of mathematics that studies the properties and applications of graphs. We will give examples to show how graphs can be used as a model in the context of using graph models to determine whether it is possible to walk all the streets of a city without going down a street twice, or whether a circuit can be implemented on a flat circuit domain.

Graphs with weights assigned to their edges can be used to solve problems such as finding the shortest path between two cities in a network. We will also examine the complexity of Euler and Hamilton paths. Now we start discussion on Euler and Hamilton' path.

Paths: Before we consider the Euler and Hamilton paths, let us explain the path itself. Suppose we have a city, we know that a city has a certain area, we consider the vertices of this area as cities and the edges as roads, a path is a distance that starts in a city, passes through several cities, and ends in a city [3].

In [1] a path ∝ in a graph G, with origin v0 and endpoint vn, is an alternating sequence of n + 1 vertices and n edges of the form

v0, e1, v1, e2, v2. . . en – 1, vn – 1, en, vn

Where each edge ei is located at the vertices vi – 1 and vi. And the n edges of a graph are called path lengths.

We show the path ∝ by a sequence of its edges ∝ = (e1, e2. . . en – 1, en ) or by a sequence of its vertices ∝ = (v0, v1, v2, . . . , , vn – 1, vn) [2].

**Theorem**: There is a path from a vertex u to a vertex v if and only if there exists a simple path from u to v [4]. For more discussion, we describe what a simple path is.

**Simple path**: A path ∝ = (v0, v1, v2, . . . , , vn – 1, vn) is simple if all its vertices are distinct. Or a path is called simple if none of its edges is repeated [1, 6,7 ].

The path ∝ = (v0, v1, v2, . . . , , vn – 1, vn) is closed if v0 = vn, i.e. its origin (∝) is equal to the limit (∝). A path is called a circuit if the path is closed and all vertices except v0 = vn are distinct. A circuit of length k is called a k-circuit.

Note: A circuit in a graph G must have length equal to or greater than three [8].

In general, as it is briefly stated in [6], If u and v are vertices in the graph G, then the distance between u and v is represented by d(u, v).

**Euler's Paths and Circuits**: Now we are going to show how Euler path look like, for a better understanding we are going to describe with a historical representation of the path.

Here as it is mentioned in [6], the subject of graph theory began in the year 1736 when the great mathematician Leonhard Euler published a paper giving the solution to its puzzle i.e. Konigsberg Bridges.

**Konigsberg Bridges**: Here we consider a city, which is the city of Konigsberg, which in the 18th century in East Prussia consisted of two islands and seven bridges. Here a question arises as follows: a person wants to walk through the city and, starting from any point in the city, cross all seven bridges but do not cross any bridge twice and reach another point in the city? The people of Konigsberg wrote a letter to the famous Swiss mathematician Euler about this question. In 1376, Euler proved that such a walk is impossible.

He replaced the islands and the two sides of the river with points and the bridges with curves and showed it in the figure below.

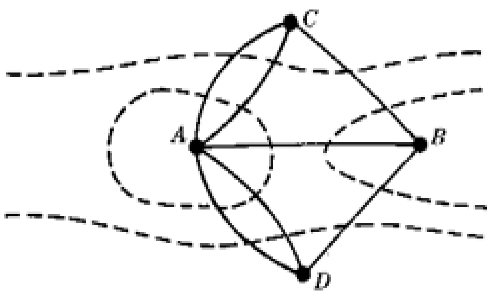



**Figure:** Graphical representation of Euler Path

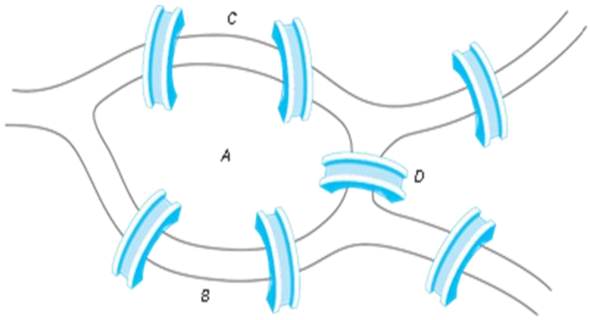



**Figure:** Bridges of Konigsberg 1786

It is easy to see that such a tour of the city of Konigsberg is possible only if the multigraph of form (b) is traversable. But this multigraph has four odd vertices and is therefore non-traversable. Thus, a pedestrian cannot circumnavigate the city by crossing each bridge only once [2, 6].

- A graph has an Euler circuit if and only if the degree of every vertex is even.
- A graph has an Euler path if and only if there are at most two vertices with odd degree

Since the bridges of Konigsberg graph has all four vertices with odd degree, there is no Euler path through the graph. Thus there is no way for the townspeople to cross every bridge exactly once [5].

**Theorem** (Konigsberg Bridge Theorem) (Euler, 1736): Let G be a connected graph. G has

An Eulerian circuit if and only if each vertex is even.

**Definition**: As stated in [2] An Euler circuit in a graph G is a simple circuit that contains every edge of G. An Euler path in a graph G is a simple path that contains every edge of G. In other words, a graph (multiple graph) G is an Eulerian graph if a traversable closed path has an Eulerian pat [7]. An Euler circuit is an Euler path which starts and stops at the same vertex [5].

More example of differential equation;

Example1: In this example, we look at the following figures and check which of the undirected graphs in Figures have an Euler circuit? And which of the graphs that are not Euler circuits have an Euler path?

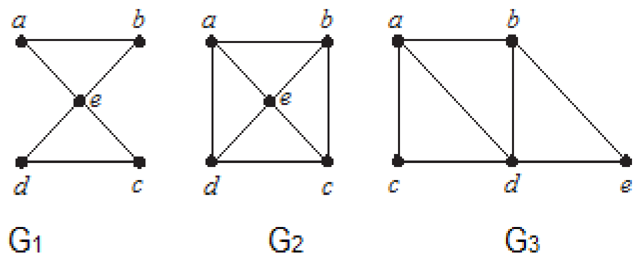



**Figure undirected graph**

**Solution**: Graph G1 has an Euler circuit, i.e., a, e, c, d, e, b, a. neither graph G2 nor G3 has an Euler circuit. However, G3 has an Euler path, i.e., a, c, d, e, b, d, a, b.

G2 does not have an Euler path.

**Example2** In this example we consider directed graph. Which of the directed graphs in bellow Figure have an Euler circuit? Of those that do not, which have an Euler path?

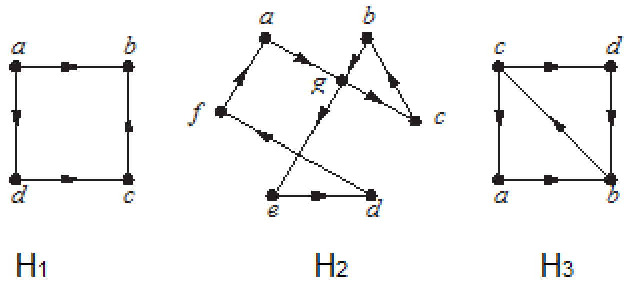



**Figure: Directed graph**

Solution: Graph H2 has an Euler circuit, i.e., a, g, c, b, g, e, d, f, a. Neither H1 nor H3 has an Euler circuit. H3 has an Euler path, i.e., c, a, b, c, d, b, but H1 does not.

**Euler's Theorem**: A finite connected graph G is Eulerian if and only if the degree of every vertex is even [2].

Now we will discuss the issue of which criterion we should consider to know in advance whether a graph is an Euler circuit, path or not. There are simple criteria for determining whether a multigraph has an Euler circuit or an Euler path. Euler discovered them when he solved the famous Konigsberg Bridge problem. We assume that all graphs discussed in this section have a finite number of vertices and edges.

What can we say if a connected multigraph has an Euler circuit? What we can show is that each vertex must have even degree. To do this, first note that an Euler circuit starts with a vertex a and continues by connecting an edge to a, say {a, b). The edge {a, b} contributes one unit to the degree (a). Each time the circuit passes through a vertex, it contributes two units to the degree of the vertex, because the circuit enters through one edge with this vertex and exits through another edge. Finally, the circuit ends where it started, contributing one unit to the degree (a). Therefore, deg(a) must be even, because the circuit contributes one unit when it starts, one unit when it ends, and two units each time it passes through a (if it does). A vertex other than a has even degree because the circuit contributes two degrees to its degree each time it passes through a vertex. We conclude that if a connected graph has an Euler circuit, then every vertex must have even degree.

Is this necessary condition also sufficient for an Euler circuit to exist? That is, if all vertices have even degree, must an Euler circuit exist in a connected multiple graph? This question can be answered positively by a construction [2].

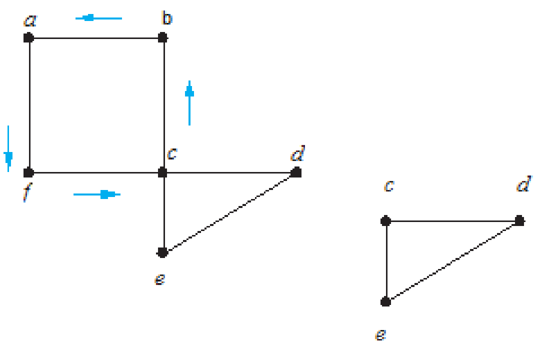



**Figure:** Formation of a Euler circuit in graph G

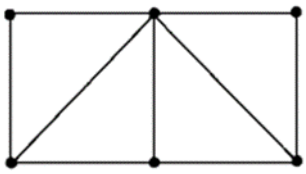
**Hamilton's Path and Circuit:**

The discussion above about Eulerian graphs emphasized edges of travel, here we focus on visiting vertices. As stated in [1] if every vertex has degree at least | V | /2, then the graph must be Hamiltonian. A Hamiltonian circuit or cycle in a graph G, named after the 19th-century Irish mathematician William Hamilton (1805–1865), is a closed path that visits each vertex in G exactly once. (Such a closed path must be a cycle). If G admits a Hamiltonian circuit, G is called a Hamiltonian graph.

Note that an Eulerian circuit traverses each edge exactly once, but may repeat vertices, while a Hamiltonian circuit visits each vertex exactly once, but may repeat edges. We have established necessary and sufficient conditions for the existence of paths and circuits that include each edge of a multigraph exactly once. Can we do the same for simple paths and circuits that visit each vertex of the graph exactly once? In honor of Hamilton, we call a cycle in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice, a Hamiltonian cycle [2, 3].

**Definition**: A simple path in a graph G, or Hamiltonian graph, is a graph with a closed path that visits each vertex exactly once. Such a path is a cycle, called a Hamiltonian cycle. Note that an Eulerian cycle visits each edge only once, but a vertex may be repeated, but a Hamiltonian cycle visits each vertex only once (except the initial and terminal vertices), but the edges may not [2].

To be noticed, A Hamiltonian graph cannot contain a vertex of degree zero or one [1]. A simple circuit in a graph G that visits each vertex exactly once is called a Hamiltonian circuit [2].



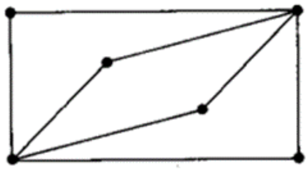
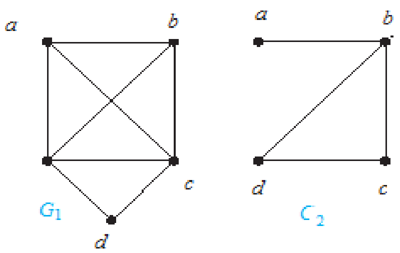
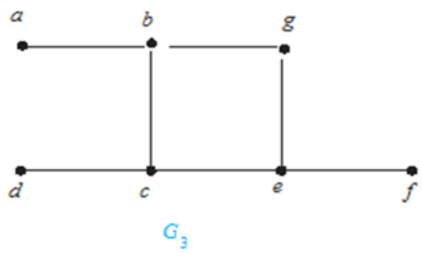


**Figure a :** Homilton's circle **Figure b :** Eular's circle

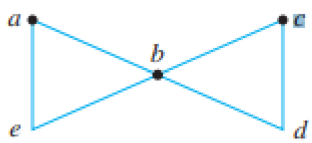
**Example**: Now look at figure below, which of the simple graphs has a Hamiltonian circuit or a Hamiltonian path?

Solution: G1 has a Hamiltonian circuit, a, b, c, d, e, a. But G2 has a Hamiltonian path, that is, a, b, c, d, and does not form a circuit. G3 It has neither a Hamiltonian circuit nor a Hamiltonian path, because any path that includes all vertices must have one of the edges {a, b}, {e, f}, and {c, d} more than once.




Example: showing that the graph G does not have a Hamiltonian circuit.




As stated in [8], We solve this example by considering the properties of the subgraph which states as follows:

i) A subgraph H (V ′, E′) of G(V , E) is called the subgraph induced by its vertices V ′ if its edge set E′ contains all edges in G whose endpoints belong to vertices in H .

(ii) If v is a vertex in G, then G − v is the subgraph of G obtained by deleting v from G and deleting all edges in G which contain v.

(iii) If e is an edge in G, then G − e is the subgraph of G obtained by simply deleting the edge e from G.

Suppose there is a connected subgraph H of G such that H has five vertices (a, b, c, d, and e) and five edges and such that every vertex of H has degree 2. Since the degree of b in G is 4 and every vertex of H has degree 2, two edges incident on b must be removed from G to create H. Edge {a, b} cannot be removed because if it were, vertex a would have degree less than 2 in H. Similar reasoning shows that edges {e, b}, {b, a}, and {b, d} cannot be removed either. It follows that the degree of b in H must be 4, which contradicts the condition that every vertex in H has degree 2 in H. Hence no such subgraph H exists, and so G does not have a Hamiltonian circuit [6].

**Theorem**: by [7], Let G be a connected graph with n vertices. Then G is Hamiltonian if $n \geq 3$ and $n \leq \deg(v)$ for each vertex v in G.

## CONCLUSION

In this article, we described Euler's and Hamilton's paths with the help of Graph formation, and at the same time, variable examples were presented to better understand the mentioned Topic. What was received from this article is that Euler's path only focused on edges that is Eulerian circuit traverses each edge exactly once, but may repeat vertices, while a Hamiltonian circuit visits each vertex exactly once.

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